

THE POLYNOMIAL REPRESENTATION OF THE TYPE A_{n-1} RATIONAL CHEREDNIK ALGEBRA IN CHARACTERISTIC $p \mid n$

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ABSTRACT. We study the polynomial representation of the rational Cherednik algebra of type A_{n-1} with generic parameter in characteristic p for $p \mid n$. We give explicit formulas for generators for the maximal proper graded submodule, show that they cut out a complete intersection, and thus compute the Hilbert series of the irreducible quotient. Our methods are motivated by taking characteristic p analogues of existing characteristic 0 results.

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1. INTRODUCTION

The present work presents a detailed study of the polynomial representation of the type A_{n-1} rational Cherednik algebra over a field of characteristic p dividing n . Rational Cherednik algebras were introduced by Etingof-Ginzburg in [EG02] as a rational degeneration of the double affine Hecke algebra dependent on two parameters \hbar and c . In characteristic 0, their type A representation theory has been the subject of extensive study. We refer the reader to [EM10] for a survey of these results.

In characteristic p and especially in the modular case, much less is known about the representation theory of the rational Cherednik algebra. In this paper, we consider the modular case $p \mid n$. For $\hbar = 1$ and generic c , we provide a complete characterization of the irreducible quotient of the polynomial representation. We give explicit generators for the unique maximal proper graded submodule J_c , show that the irreducible quotient is a complete intersection, and compute its Hilbert series.

Our techniques are inspired by taking characteristic p analogues of results about Cherednik algebras in characteristic 0. In particular, our explicit expression for generators of J_c was obtained by converting expressions with complex residues to equivalent expressions dealing only with formal power series which may be interpreted in characteristic p . While we restrict our study to the polynomial representation in type A , we view it as a test case for this philosophy, which we believe may admit wider application.

We now state our results precisely and explain their relation to other recent work.

1.1. The rational Cherednik algebra in positive characteristic. We work over an algebraically closed field k of characteristic $p > 0$ and fix n so that $p \mid n$. Let S_n denote the symmetric group on n elements, $V = k^n$ its permutation representation, and $s_{ij} \in S_n$ the transposition permuting i and j . Fix a basis y_1, \dots, y_n for V and a dual basis x_1, \dots, x_n for V^* . Let \mathfrak{h} and \mathfrak{h}^* be the dual $(n-1)$ -dimensional S_n -representations which are subrepresentation and quotient of V and V^* , respectively given by

$$\mathfrak{h} = \text{span}\{y_i - y_j \mid i \neq j\} \text{ and } \mathfrak{h}^* = V^*/(x_1 + \dots + x_n).$$

The action of S_n on \mathfrak{h} and \mathfrak{h}^* is given explicitly by natural permutation of basis vectors.

Fix constants \hbar and c in k . Denoting the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$ by $T(\mathfrak{h} \oplus \mathfrak{h}^*)$, the *type A_{n-1} rational Cherednik algebra* $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ is the quotient of $k[S_n] \rtimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x_i, x_j] = 0, \quad [y_i - y_j, y_l - y_m] = 0, \quad [y_i - y_j, x_i] = \hbar - cs_{ij} - c \sum_{t \neq i} s_{it}, \quad [y_i - y_j, x_l] = cs_{il} - cs_{jl}$$

for all $1 \leq i, j, l, m \leq n$ such that i, j, l are distinct and $l \neq m$. There is a \mathbb{Z} -grading on $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ given by setting $\deg x = 1$ for $x \in \mathfrak{h}^*$, $\deg y = -1$ for $y \in \mathfrak{h}$, and $\deg g = 0$ for $g \in k[S_n]$. In addition, $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ admits a PBW decomposition

$$\mathcal{H}_{\hbar,c}(\mathfrak{h}) \simeq \text{Sym}(\mathfrak{h}) \otimes_k k[S_n] \otimes_k \text{Sym}(\mathfrak{h}^*).$$

For any $\alpha \neq 0$, $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ and $\mathcal{H}_{\alpha\hbar,\alpha c}(\mathfrak{h})$ are isomorphic as algebras, so only the cases $\hbar = 0$ or $\hbar = 1$ need be considered. In this paper, we restrict our attention to $\hbar = 1$.

1.2. Polynomial representation of the rational Cherednik algebra. The rational Cherednik algebra $\mathcal{H}_{1,c}(\mathfrak{h})$ admits a $\mathbb{Z}_{\geq 0}$ -graded representation on the polynomial ring $A = \text{Sym}(\mathfrak{h}^*)$, known as the *polynomial representation*. The actions of $\text{Sym}(\mathfrak{h}^*)$ and $k[S_n]$ on A are by left multiplication and the S_n action on \mathfrak{h}^* , respectively. The action of $\text{Sym}(\mathfrak{h})$ is implemented by letting $y \in \mathfrak{h}$ act by the *Dunkl operator*

$$D_y = \partial_y - \sum_{m < l} c \langle y, x_m - x_l \rangle \frac{1 - s_{ml}}{x_m - x_l},$$

where we note that $\frac{1-s_{ml}}{x_m-x_l}f$ is a polynomial for $f \in A$. Explicitly, we have

$$D_{y_i - y_j} = \partial_{y_i - y_j} - c \sum_{m \neq i} \frac{1 - s_{mi}}{x_i - x_m} + c \sum_{m \neq j} \frac{1 - s_{mj}}{x_j - x_m},$$

where $\partial_{y_i - y_j}$ is the differential operator satisfying $\partial_{y_i - y_j}(x) = \langle y_i - y_j, x \rangle$ for all $x \in \mathfrak{h}^*$.

1.3. Maximal proper graded submodule and irreducible quotient of polynomial representation.

As described in [BC13, Section 2.5], there is a contravariant form

$$\beta_c : \text{Sym}(\mathfrak{h}^*) \otimes \text{Sym}(\mathfrak{h}) \rightarrow k$$

defined by setting $\beta_c(1, 1) = 1$ and imposing for all $x \in \mathfrak{h}^*, y \in \mathfrak{h}, f \in \text{Sym}(\mathfrak{h}^*), g \in \text{Sym}(\mathfrak{h})$ that

$$\beta_c(xf, g) = \beta_c(f, D_x(g)) \quad \text{and} \quad \beta_c(f, yg) = \beta_c(D_y(f), g).$$

where for $x \in \mathfrak{h}^*$ we denote by D_x the Dunkl operator implementing the action of $\mathcal{H}_{1,c}(\mathfrak{h}^*)$ on its polynomial representation $\text{Sym}(\mathfrak{h})$.

The polynomial representation $\text{Sym}(\mathfrak{h}^*)$ has unique maximal graded proper submodule $J_c = \ker(\beta_c)$. By the definition of β_c , J_c contains the ideal generated by all homogeneous vectors $f \in A$ of positive degree that lie in the kernel of all Dunkl operators D_y . Such f are known as *singular vectors*. The quotient $L = A/J_c$ is an irreducible representation of $\mathcal{H}_{1,c}(\mathfrak{h})$. It inherits a $\mathbb{Z}_{\geq 0}$ -grading from A , and for L_j the degree j subspace of L , we may define its Hilbert series as

$$h_L(t) = \sum_{j \geq 0} \dim L_j t^j.$$

1.4. Statement of the main result. For a formal power series $r(z)$, we denote by $[z^l]r(z)$ the coefficient of z^l in $r(z)$. Throughout the paper, we will consider formal power series in z considered as expansions of rational functions around $z = 0$. For $i = 1, \dots, n-1$, define the formal power series

$$F_i(z) = \frac{1}{1 - x_i z} \sum_{m=0}^{p-1} \binom{c}{m} \left(\prod_{j=1}^n (1 - x_j z) - 1 \right)^m$$

for $\binom{c}{m} = \frac{c(c-1)\cdots(c-m+1)}{m!}$. Denote by f_i the coefficients $f_i = [z^p]F_i(z)$.

Theorem 4.1. For generic c , f_1, \dots, f_{n-1} are linearly independent and generate the maximal proper graded submodule J_c of the polynomial representation for $\mathcal{H}_{1,c}(\mathfrak{h})$. The irreducible quotient $L = A/J_c$ is a complete intersection with Hilbert series

$$h_L(t) = \left(\frac{1 - t^p}{1 - t} \right)^{n-1}.$$

Remark. In Theorem 4.1, by generic c we mean c avoiding finitely many values.

1.5. Connections to previous work. Our study is motivated by previous work on the representation theory of the type A rational Cherednik algebra in both characteristic 0 and p . The type A non-modular case $p \gg n$ was studied in [BFG06], and some properties of the maximal proper graded submodule of the polynomial representation were given in both modular and non-modular cases in [BC13]. In the modular case $p \mid n$, for $p = 2$ the polynomial representation associated to the n -dimensional permutation representation was studied in [Lia12].

Theorem 1.1 ([Lia12, Theorem 5.1]). The irreducible quotient of the polynomial representation associated to the n -dimensional permutation representation is a complete intersection with Hilbert series

$$h(t) = (1+t)^n(1+t^2).$$

The corresponding maximal proper graded submodule is generated by $n-1$ elements of degree 2 and one element of degree 4.

It was further conjectured by Lian in [Lia12, Conjecture 5.2] that for all p the corresponding irreducible is a complete intersection with J_c having $n-1$ generators in degree p and a single generator in degree p^2 . Our results are consistent with the restriction of Lian's conjecture to the case when \mathfrak{h} is the $(n-1)$ -dimensional quotient. It would be interesting to extend our work to prove Lian's conjecture in full. For general $p \mid n$, a submodule of the maximal proper graded submodule was computed in [DS14, Proposition 6.1].

In characteristic 0, our results parallel the explicit decomposition of the polynomial representation of the type A rational Cherednik algebra given in [BEG03, CE03]. There, the polynomial representation is irreducible unless $c = \frac{r}{n}$ for some integer r , and an explicit set of generators of the maximal proper graded submodule is known.

Proposition 1.2 ([CE03, Proposition 3.1]). If $\text{char}(k) = 0$ and $c = \frac{r}{n}$, the maximal proper graded submodule $J_c \subset A$ of the polynomial representation A of $\mathcal{H}_{1,c}(\mathfrak{h})$ is generated by

$$\text{Res}_\infty \left[\frac{dz}{z-x_j} \prod_{i=1}^n (z-x_i)^c \right] \text{ for } j = 1, \dots, n-1.$$

We interpret the characteristic p analogue of Proposition 1.2 to mean that if $r = p$ and $p \mid n$, then since p/n is equivalent to $0/0$ and thus an indeterminacy in characteristic p , taking $c = p/n$ in characteristic 0 should correspond to taking c generic in characteristic p . While this substitution is of course invalid, Proposition 1.2 may be interpreted as a statement about certain formal power series. By using a power series version of this construction of generators which makes sense in characteristic p , we are able to mimic the arguments of [BEG03, CE03] to show that they cut out a complete intersection and generate the entire ideal. We believe that the philosophy of taking characteristic p analogues of characteristic 0 results for the rational Cherednik algebra should apply more generally and hope to explore this further in future work.

1.6. Outline of the paper. The remainder of this paper is organized as follows. In Section 2, we check that the generators f_1, \dots, f_{n-1} are linearly independent singular vectors. In Section 3, we show that they cut out a complete intersection. In Section 4, we put these facts together to conclude Theorem 4.1.

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2. AN EXPLICIT CONSTRUCTION OF SINGULAR VECTORS

2.1. Definition of the singular vectors. In A , define the polynomials

$$g(z) = \prod_{j=1}^n (1 - x_j z) \quad \text{and} \quad F(z) = \sum_{m=0}^{p-1} \binom{c}{m} (g(z) - 1)^m.$$

In these terms, we have $F_i(z) = \frac{F(z)}{1-x_i z}$ and $f_i = [z^p] \frac{F(z)}{1-x_i z}$. We will show that f_i are singular vectors.

2.2. Computation of some partial derivatives. We begin by computing some partial derivatives of F which will be useful for computing the action of the Dunkl operators.

Lemma 2.1. We have $[z^0]g(z) = 1$ and $[z^1]g(z) = 0$, meaning $z^2 \mid g(z) - 1$.

Proof. For elementary symmetric polynomials e_2, \dots, e_n , we have the expansion

$$g(z) = \prod_{j=1}^n (1 - x_j z) = 1 - z \sum_i x_i + z^2 e_2(x_1, \dots, x_n) + \dots + (-1)^n z^n e_n(x_1, \dots, x_n).$$

Recalling that $\sum_i x_i = 0$ in A , we see that $[z^1]g(z) = 0$ and $[z^0]g(z) = 1$, so $z^2 \mid g(z) - 1$ as desired. \square

Lemma 2.2. For some formal power series $V(z)$ with $[z^l]V(z) = 0$ for $l = 0, \dots, p-1$, we have

$$F'(z) = V(z) - \sum_{j=1}^n \frac{cx_j}{1 - x_j z} F(z).$$

Proof. We see easily that $\frac{\partial g}{\partial z} = -g(z) \sum_j \frac{x_j}{1 - x_j z}$. We now consider $\frac{\partial F}{\partial z}$. We compute

$$\begin{aligned} \frac{\partial F}{\partial z} &= \sum_{m=1}^{p-1} m \binom{c}{m} (g(z) - 1)^{m-1} \frac{\partial g}{\partial z} \\ &= - \sum_j \frac{x_j}{1 - x_j z} \sum_{m=0}^{p-2} c \binom{c-1}{m} (g(z) - 1)^m (g(z) - 1 + 1) \\ &= - \sum_j \frac{x_j}{1 - x_j z} \left(\sum_{m=0}^{p-2} c \binom{c-1}{m} (g(z) - 1)^m + \sum_{m=1}^{p-1} c \binom{c-1}{m-1} (g(z) - 1)^m \right) \\ &= - \sum_j \frac{x_j}{1 - x_j z} \left(\sum_{m=0}^{p-1} c \binom{c}{m} (g(z) - 1)^m - c \binom{c-1}{p-1} (g(z) - 1)^{p-1} \right) \\ &= - \sum_j \frac{cx_j}{1 - x_j z} F(z) + \sum_j \frac{x_j}{1 - x_j z} c \binom{c-1}{p-1} (g(z) - 1)^{p-1}. \end{aligned}$$

Defining the formal power series

$$V(z) = \sum_j \frac{x_j}{1 - x_j z} c \binom{c-1}{p-1} (g(z) - 1)^{p-1},$$

we see that $F'(z) = V(z) - \sum_{j=1}^n \frac{cx_j}{1 - x_j z} F(z)$. It remains only to show that $[z^l]V(z) = 0$ for $l = 0, \dots, p-1$, which follows by noting that $(g(z) - 1)^{p-1} \mid V(z)$, applying Lemma 2.1, and noting $p \geq 2$. \square

Lemma 2.3. For some formal power series $G(z)$ with $[z^l]G(z) = 0$ for $l = 0, \dots, p$, we have

$$\partial_{y_2 - y_1}(F(z)) = G(z) - \left(\frac{zc}{1 - x_2 z} - \frac{zc}{1 - x_1 z} \right) F(z).$$

Proof. We may compute $\partial_{y_2 - y_1}(g(z)) = g(z) \left(-\frac{z}{1 - x_2 z} + \frac{z}{1 - x_1 z} \right)$. Using this, we see that

$$\begin{aligned} \partial_{y_2 - y_1}(F(z)) &= \left(\sum_{m=1}^{p-1} m \binom{c}{m} (g(z) - 1)^{m-1} \right) \partial_{y_2 - y_1}(g(z)) \\ &= \left(-\frac{z}{1 - x_2 z} + \frac{z}{1 - x_1 z} \right) \left(\sum_{m=1}^{p-1} m \binom{c}{m} (g(z) - 1)^{m-1} \right) g(z) \\ &= \left(-\frac{z}{1 - x_2 z} + \frac{z}{1 - x_1 z} \right) \left(\sum_{m=0}^{p-2} c \binom{c-1}{m} (g(z) - 1)^m + \sum_{m=0}^{p-2} c \binom{c-1}{m} (g(z) - 1)^{m+1} \right) \\ &= \left(-\frac{zc}{1 - x_2 z} + \frac{zc}{1 - x_1 z} \right) \left(F(z) - \binom{c-1}{p-1} (g(z) - 1)^{p-1} \right). \end{aligned}$$

Defining $G(z) = \left(\frac{zc}{1-x_2z} - \frac{zc}{1-x_1z} \right) \binom{c-1}{p-1} (g(z) - 1)^{p-1}$, we have shown that

$$\partial_{y_2-y_1}(F(z)) = G(z) - \left(\frac{zc}{1-x_2z} - \frac{zc}{1-x_1z} \right) F(z)$$

It remains only to show that $[z^l]G(z) = 0$ for $l = 0, \dots, p$, which follows by noting that $z(g(z) - 1)^{p-1} \mid G(z)$, applying Lemma 2.1, and noting $p \geq 2$. \square

2.3. Proving f_1, \dots, f_{n-1} are singular vectors.

Proposition 2.4. The elements f_1, \dots, f_{n-1} are singular vectors in A .

Proof. We must show that for $i = 1, \dots, n-1$, f_i is annihilated by $D_{y_j-y_l}$ for all $j \neq l$. First, by symmetry it suffices to consider f_1 . Because the Dunkl operators $D_{y_i-y_j}$ for all $i \neq j$ are spanned by $D_{y_i-y_1}$ for $1 < i \leq n$, it suffices to show $D_{y_i-y_1}f_1 = 0$. Finally, because f_1 is symmetric in the x_i for $i > 1$, it suffices to show that $D_{y_2-y_1}f_1 = 0$.

Recall by Lemma 2.3 that $\partial_{y_2-y_1}(F(z)) = G(z) - \left(\frac{zc}{1-x_2z} - \frac{zc}{1-x_1z} \right) F(z)$ for a power series $G(z)$ with $[z^l]G(z) = 0$ for $l = 0, \dots, p$. In terms of $G(z)$, we can calculate $\partial_{y_2-y_1}(F_1(z))$ as

$$\begin{aligned} \partial_{y_2-y_1}(F_1(z)) &= -\frac{z}{(1-x_1z)^2}F(z) + \frac{1}{1-x_1z}\partial_{y_2-y_1}(F(z)) \\ &= -\frac{z}{1-x_1z}F_1(z) + \frac{1}{1-x_1z}\left(\frac{zc}{1-x_1z} - \frac{zc}{1-x_2z}\right)F(z) + \frac{G(z)}{1-x_1z} \\ &= \left(\frac{z(c-1)}{1-x_1z} - \frac{zc}{1-x_2z}\right)F_1(z) + \frac{G(z)}{1-x_1z}. \end{aligned}$$

In addition, we have that

$$\frac{1-s_{1i}}{x_1-x_i}(F_1(z)) = \frac{1}{x_1-x_i}\left(\frac{1}{1-x_1z} - \frac{1}{1-x_iz}\right)F(z) = \frac{z}{(1-x_iz)(1-x_1z)}F(z) = \frac{z}{1-x_iz}F_1(z).$$

Finally, by Lemma 2.2, we have

$$\frac{\partial F}{\partial z} = V(z) - \sum_j \frac{cx_j}{1-x_jz}F(z),$$

where $[z^l]V(z) = 0$ for $l = 0, \dots, p-1$. From this, it follows that

$$\begin{aligned} \frac{\partial F_1}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{F(z)}{1-x_1z}\right) = \frac{1}{1-x_1z}\frac{\partial F}{\partial z} + \frac{x_1}{(1-x_1z)^2}F(z) \\ &= \frac{V(z)}{1-x_1z} - \frac{1}{1-x_1z}\sum_j \frac{cx_j}{1-x_jz}F(z) + \frac{x_1}{(1-x_1z)^2}F(z) \\ &= \frac{V(z)}{1-x_1z} - \sum_j \frac{cx_j}{1-x_jz}F_1(z) + \frac{x_1}{1-x_1z}F_1(z). \end{aligned}$$

Because $F_1(z)$ is invariant under s_{ij} for $i, j > 1$, we now compute

$$\begin{aligned}
D_{y_2-y_1}(F_1(z)) &= \left(\partial_{y_2-y_1} - c \frac{1-s_{12}}{x_2-x_1} + c \sum_{j>1} \frac{1-s_{1j}}{x_1-x_j} \right) F_1(z) \\
&= \partial_{y_2-y_1}(F_1(z)) - c \frac{1-s_{12}}{x_2-x_1} F_1(z) + c \sum_{j>1} \frac{1-s_{1j}}{x_1-x_j} F_1(z) \\
&= \frac{G(z)}{1-x_1z} + \left(\frac{z(c-1)}{1-x_1z} - \frac{zc}{1-x_2z} \right) F_1(z) + \frac{zc}{1-x_2z} F_1(z) + \sum_{j>1} \frac{zc}{1-x_jz} F_1(z) \\
&= \frac{G(z)}{1-x_1z} - \frac{z}{1-x_1z} F_1(z) + \sum_j \frac{zc}{1-x_jz} F_1(z) \\
&= \frac{G(z)}{1-x_1z} - zF_1(z) + zF_1(z) - \frac{z}{1-x_1z} F_1(z) + \sum_j \left(\frac{zc}{1-x_jz} - zc \right) F_1(z) \\
&= \frac{G(z)}{1-x_1z} - zF_1(z) - \frac{x_1z^2}{1-x_1z} F_1(z) + \sum_j \frac{x_jcz^2}{1-x_jz} F_1(z) \\
&= \frac{G(z)}{1-x_1z} - zF_1(z) - z^2 \frac{\partial F_1}{\partial z} + z^2 \frac{V(z)}{1-x_1z} \\
&= \frac{G(z) + z^2V(z)}{1-x_1z} - zF_1(z) - z^2 \frac{\partial F_1}{\partial z},
\end{aligned}$$

where in the fifth step we have subtracted $nzczF_1(z)$. We note that $[z^p] \frac{G(z)+z^2V(z)}{1-x_1z}$ is a linear combination of $[z^l](G(z) + z^2V(z))$ for $0 \leq l \leq p$, hence a linear combination of $[z^l]G(z)$ for $0 \leq l \leq p$ and $[z^l]V(z)$ for $0 \leq l \leq p-2$. By Lemmas 2.2 and 2.3, these coefficients of $G(z)$ and $V(z)$ are all 0, hence $[z^p] \frac{G(z)+z^2V(z)}{1-x_1z} = 0$. We conclude that

$$[z^p]D_{y_2-y_1}(F_1(z)) = [z^p] \left(-zF_1(z) - z^2 \frac{\partial F_1}{\partial z} \right).$$

If $b = [z^{p-1}](F_1(z))$, then $[z^p](-zF_1(z)) = -b$ and $[z^p](-z^2 \frac{\partial F_1}{\partial z}) = b$, which implies that

$$D_{y_2-y_1}f_1 = [z^p]D_{y_2-y_1}(F_1(z)) = [z^p] \left(-zF_1(z) - z^2 \frac{\partial F_1}{\partial z} \right) = -b + b = 0. \quad \square$$

2.4. Proof of linear independence of f_1, \dots, f_{n-1} .

Proposition 2.5. For generic c , f_1, \dots, f_{n-1} are linearly independent degree p homogeneous polynomials.

Proof. We have the expansion

$$F_i(z) = \frac{1}{1-x_iz} \sum_{m=0}^{p-1} \binom{c}{m} (g(z)-1)^m = \sum_{l=0}^{\infty} x_i^l z^l \sum_{m=0}^{p-1} \binom{c}{m} (g(z)-1)^m.$$

Because for any l the coefficient of z^l in each factor is a homogeneous polynomial of degree l , we see that $[z^p]F_i(z)$ is homogeneous of degree p .

For linear independence, suppose that $\sum_{i=1}^{n-1} \lambda_i f_i = 0$ for some $\lambda_i \in k$. Substitute $x_n = -1$, $x_j = 1$ and $x_i = 0$ for $i \neq j, i < n$ so that $g(z) = (1-z)(1+z) = 1-z^2$ and hence

$$F_j(z) = \sum_{l=0}^{\infty} z^l \sum_{m=0}^{p-1} \binom{c}{m} (-z^2)^m$$

and

$$F_i(z) = \sum_{l=0}^{\infty} 0^l z^l \sum_{m=0}^{p-1} \binom{c}{m} (-z^2)^m = \sum_{m=0}^{p-1} \binom{c}{m} (-z^2)^m \text{ for } i < n-1, i \neq j.$$

If $p = 2$, we see that $[z^2]F_j(z) = 1 - c$ and $[z^2]F_i(z) = -c$, so varying j implies that

$$\lambda_j = c \sum_{i=1}^{n-1} \lambda_i \text{ for all } j.$$

In particular, all λ_i have common value $\lambda \in k$ solving $(1 - c(n-1))\lambda = 0$, which for $c \neq -1$ and hence for c generic implies that $\lambda = 0$, giving linear independence.

If $p > 2$, we have

$$[z^p]F_j(z) = f_j = \sum_{m=0}^{(p-1)/2} (-1)^m \binom{c}{m} = \binom{c-1}{(p-1)/2}$$

and $[z^p]F_i(z) = f_i = 0$ for $i < n$ and $i \neq j$. For $c \notin \{1, 2, \dots, (p-1)/2\}$ and hence for generic c , we have $\binom{c-1}{(p-1)/2} \neq 0$, meaning that

$$\sum_{i=1}^{n-1} \lambda_i f_i = \lambda_j \binom{c-1}{(p-1)/2} = 0,$$

which implies $\lambda_j = 0$. Varying j implies that $\lambda_j = 0$ for all j , again yielding linear independence. \square

3. COMPLETE INTERSECTION PROPERTIES

Consider the homogeneous ideal $I_c = \langle f_1, \dots, f_{n-1} \rangle \subset A$ generated by the f_i . In this section, we will show that A/I_c is a complete intersection. Recall from [Har92, Example 11.8] that for a homogeneous ideal $I \subset k[X_0, \dots, X_m]$ with a minimal set of generators of size l , the quotient ring $k[X_0, \dots, X_m]/I$ is a complete intersection if the closed projective subvariety of \mathbb{P}^m defined by I has dimension $m - l$.

Proposition 3.1. For generic c , the quotient A/I_c is a complete intersection.

Proof. By Proposition 2.5, I_c has a set of $n - 1$ linearly independent and therefore minimal generators f_1, \dots, f_{n-1} in degree p . Therefore A/I_c is a complete intersection if and only if I_c cuts out a projective variety of dimension $(n - 2) - (n - 1) = -1$, meaning it is empty. This occurs if and only if the saturation of I_c is A , which occurs if and only if $\dim_k(A/I_c) < \infty$.

If $c = 0$, by definition of f_i we see that $f_i = x_i^p$. Note that

$$f_n = x_n^p = (-x_1 - \dots - x_{n-1})^p = -x_1^p - \dots - x_{n-1}^p = -f_1 - \dots - f_{n-1} \in I_0,$$

meaning that $(x_1^p, \dots, x_n^p) \subset I_0$ and hence $\dim_k(A/I_0) < \infty$. Now, let A^d and I_c^d denote the degree d pieces of A and I_c , respectively. Choose a monomial basis $\{t_i\}$ for A^d independent of c , and let I_c^d be spanned by a finite set of polynomials $\{g_j\}$ with $g_j = \sum_i h_{ji}(c)t_i$. Notice that $\dim I_c^d$ is given by the size of the maximal non-vanishing minor of the matrix $H = (h_{ji})$. On the other hand, we just showed that there is some degree $d > 0$ so that $I_0^d = A^d$, meaning that $\dim I_c^d < \dim A^d$ exactly when c lies in the zero set of all size $\dim A^d$ minors of H , viewed as polynomials in c . This implies that for all but finitely many c we have $\dim I_c^d = \dim A^d$, hence $\dim_k(A/I_c) < \infty$ and A/I_c is a complete intersection. \square

Remark. Proposition 3.1 is a formal power series analogue of [CE03, Theorem 3.2]. However, our proof differs from the “residues by parts” argument which appears there, as the crucial [CE03, Lemma 3.2] fails in the modular case. It is interesting to note that our proof does not appear to translate to the characteristic 0 case, as there is no analogue for the operation of specializing c to 0.

4. PROOF OF THE MAIN RESULT

We now put everything together to obtain our main result.

Theorem 4.1. For generic c , f_1, \dots, f_{n-1} are linearly independent and generate the maximal proper graded submodule J_c of the polynomial representation for $\mathcal{H}_{1,c}(\mathfrak{h})$. The irreducible quotient $L = A/J_c$ is a complete intersection with Hilbert series

$$h_L(t) = \left(\frac{1 - t^p}{1 - t} \right)^{n-1}.$$

Proof. By [BC13, Proposition 3.4], the Hilbert series of L is

$$h_L(t) = \left(\frac{1-t^p}{1-t} \right)^{n-1} h(t^p)$$

for a polynomial $h(t)$ with nonnegative integer coefficients. On the other hand, by Propositions 2.5 and 3.1, A/I_c is a complete intersection with $n-1$ linearly independent degree p generators f_1, \dots, f_{n-1} . Its Hilbert series is

$$h_{A/I_c}(t) = \left(\frac{1-t^p}{1-t} \right)^{n-1}.$$

By Proposition 2.4, the generators f_1, \dots, f_{n-1} of I_c are singular vectors, so $I_c \subseteq J_c$, implying that $h_{A/I_c}(t) \geq h_{A/J_c}(t)$ coefficient-wise. We conclude that $h(t) = 1$, hence $h_{A/I_c}(t) = h_{A/J_c}(t)$ and $I_c = J_c$, completing the proof. \square

Remark. In the proof of Proposition 3.1, we require that c avoids $\{-1, 1, \dots, (p-1)/2\}$ for Proposition 2.5 and that c avoids a non-explicit finite set given by vanishing of a determinantal ideal. These are the only uses of the assumption that c is generic, so Proposition 3.1 and Theorem 4.1 hold for c avoiding these values.

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